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Limit Theorems for Transient Markov Chains*

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ABSTRACT

An investigation of the asymptotic behavior, for large n, of the quantities

$$P_x(n < V_B{}^r < \infty;\ X_{V_B{}^r} = y),$$

where $V_B{}^r$ is the time of the r-th visit to a finite, non-empty set B of states in a countable state space transient Markov chain X_n. These results are shown to yield information on the asymptotic behavior for solutions to the exterior Dirichlet problem for B, and on the error in approximating $P_x(N_n(B) \leq r)$ by $P_x[N(B) \leq r]$, where $N_n(B)$ is the occupation time in B by time n and $N(B) = \lim_n N_n(B)$. Applications are given to several specific classes of Markov chains.

1. INTRODUCTION

In this paper we continue the work begun in [2] on the asymptotic behavior of transient Markov chains. As before, our basic assumption will be that the Markov chain $\{X_n\}$, with states in a denumerable set Ω (which we may label with the integers when convenient), is irreducible

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and transient. Previously we were interested in the behavior, for large n, of the quantity $P_x(n < V_B < \infty)$, where V_B is the hitting time after time 0 for a finite set B. Here we shall be concerned with the asymptotic behavior of the more general quantities

$$P_x(n < V_B^r < \infty;\ X_{V_B^r} = y),$$

where V_B^r is the time of the r-th visit to the finite non-empty set B.

The following notation will be used throughout: For convenience set $V_B^0 = 0$ and define the successive entrance times to B as follows:

$$V_B = V_B^1, \text{ where, for any } r \geq 1,\ V_B^r = \min \{n > V_B^{r-1}: X_n \in B\};$$
$$V_B^r = \infty \quad \text{if} \quad X_n \notin B \quad \text{for all} \quad n > V_B^{r-1}, \quad \text{or if} \quad V_B^{r-1} = \infty.$$

Set

$$1_B(x) = 1 \qquad \text{if } x \in B,$$
$$= 0 \qquad \text{if } x \notin B;$$

and define the occupation time in B by time n as

$$N_n(B) = \sum_{j=1}^{n} 1_B(X_j).$$

The total occupation time in B is then

$$N(B) = \lim_{n\to\infty} N_n(B).$$

Set

$$P_x(\cdot) = P(\cdot \mid X_0 = x), \qquad E_x(\cdot) = E(\cdot \mid X_0 = x).$$

The following matrix quantities will be used.

$$
\begin{aligned}
{}_BP_n(x, y) &= P_x(X_n = y,\ V_B \geq n);\\
P_n(x, y) &= P_x(X_n = y);\\
R_n(x, y) &= \sum_{j=n+1}^{\infty} P_j(x, y), \qquad R_n = R_n(0, 0);\\
{}_BR_n(x, y) &= \sum_{j=n+1}^{\infty} {}_BP_j(x, y);\\
I_B(x, y) &= \delta_{xy} \qquad \text{if } x,\ y \in B,\\
&= 0 \text{ elsewhere;}
\end{aligned}
$$

$$\begin{aligned} \Pi_B(x, y) &= P_x(V_B < \infty;\ X_{V_B} = y) = ({}_BR_0I_B)\,(x, y); \\ H_B(x, y) &= \Pi_B(x, y) \qquad \text{if } x \notin B \\ &= \delta_{xy} \qquad\qquad\quad \text{if } x \in B. \end{aligned}$$

We will also need to use generating functions. The generating function of $\{{}_BP_n\}$ will be denoted by ${}_BP^t$, where $0 < t < 1$. Similar notation will be used for generating functions of other matrix quantities. In particular $\Pi_B{}^t = {}_BP^tI_B$.

In Lemma 3.1 of [2] it was shown that, if for some pair of states x_0, y_0 (not necessarily distinct) we knew that

$$\sum_{n=0}^{\infty} R_n(x_0, y_0) < \infty,$$

then

$$\sum_{n=0}^{\infty} R_n(x, y) < \infty$$

for all states x, y.

DEFINITION 1. An irreducible, transient chain is strongly transient if for some state 0,

$$\sum_{n=0}^{\infty} R_n(0,\ 0) < \infty.$$

The chain is weakly transient if the above series diverges.

The intuitive meaning of weak and strong transience is easy to see. If the chain is transient, then we know that if $T_B = \min\,\{n \geq 0\colon X_m \notin B$ for all $m > n\}$ (i.e., if T_B is the time of *last* visit to B), then $P_x(T_B < \infty) = 1$. As was shown in [2],

$$E_xT_B = \sum_{y \in B} \left[\sum_{n=0}^{\infty} R_n(x, y)P_y(V_B = \infty) \right].$$

Thus a chain is strongly transient if and only if $E_xT_B < \infty$ for all x and every finite B. The analog with positive and null recurrence is apparent, and (as we shall see) there is a great difference in the behavior of strongly and weakly transient chains.

DEFINITION 2. An irreducible, transient chain is regular if

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} R_k(x, y)}{\sum_{k=0}^{n} R_k(0, 0)} = A(x, y)$$

exists for all states x, y.

We observe that every strongly transient chain is regular, and that for such a chain,

$$A(x, y) = \frac{[(R_0 - I)R_0]\,(x, y)}{[(R_0 - I)R_0]\,(0, 0)}\,.$$

However, as we shall see, there are also weakly transient chains that are regular. In any case the matrix $A(x, y)$ is a superregular function of x for fixed y and a superregular measure in y for fixed x. For some weakly transient chains, $A(x, y) = \pi(y)$. In such cases the results we obtain take particularly simple forms (see Section 4).

We now briefly summarize the contents of the paper. The main result in Section 2 is that, in every regular transient chain,

$$\lim_{m\to\infty} \frac{\sum_{n=0}^{m} P_x(n < V_B{}^r < \infty;\ X_{V_B{}^r} = y)}{\sum_{n=0}^{m} R_n(0, 0)} = L_B(x, y, r), \quad (1.1)$$

where

$$L_B(x, y, r) = \left[\sum_{j=0}^{r-1} (\Pi_B)^j (I - \Pi_B) A I_B (I - \Pi_B)\,(\Pi_B)^{r-1-j}\right](x, y). \quad (1.2)$$

In particular, in every strongly transient chain,

$$\sum_{n=0}^{\infty} P_x(n < V_B{}^r < \infty; X_{V_B{}^r} = y) = \left[\sum_{j=0}^{r-1} (\Pi_B)^j\,{}_B R_0 H_B (\Pi_B)^{r-1-j}\right](x, y).$$

These results are used in Corollary 2.1 to obtain information about the rate of approach of solutions of the time-dependent Dirichlet problem for a set B to that of the corresponding time-independent solution. In Corollary 2.2 the above results are shown to yield information on the error term in approximating $P_x(N_n(B) \leq r)$ by $P_x(N(B) \leq r)$.

In Section 3 we show that, under much more stringent conditions, limits of individual terms in (1.1), rather than partial sums, exist and have the same form as (1.2), except now $A(x, y)$ is replaced by

$$D(x, y) = \lim_{n\to\infty} R_n(x, y)\, R_n(0, 0)^{-1}$$

(which is assumed to exist).

In Section 4 we apply the results in the previous two sections to some specific examples.

2. Ratio Limits

Throughout this section, B will be a finite nonempty subset of Ω, and x, y will be arbitrary states. For simplicity we set $R_n = R_n(0, 0)$.

Theorem 2.1. *In every regular chain*

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} {}_BR_j(x, y)}{\sum_{j=0}^{n} R_j} = M_B(x, y) \tag{2.1}$$

exists. Moreover,

$$M_B(x, y) = [(I - \Pi_B)A(I_B - I_B\Pi_B)](x, y), \qquad y \in B \tag{2.2}$$

$$= [(I - \Pi_B)A](x, y) - [(I - \Pi_B)AI_{BB}R_0](x, y), \qquad y \notin B \tag{2.3}$$

In particular, if the chain is strongly transient, then

$$\sum_{n=0}^{\infty} P_x(n < V_B < \infty; X_{V_B} = y) = ({}_BR_0H_B)(x, y). \tag{2.4}$$

Proof: We have

$${}_BP_n(x, y) = P_n(x, y) - \sum_{k=1}^{n-1} \sum_{z\in B} {}_BP_k(x, z)P_{n-k}(z, y).$$

Taking generating functions, we obtain the identity

$${}_BP^t = P^t - \Pi_B{}^tP^t. \tag{2.5}$$

A simple computation now gives

$${}_BR^t + {}_BR^tI_BR_0 = (I - \Pi_B{}^t)R^t, \tag{2.6}$$

and, consequently,

$${}_BR^t(I_B + I_BR_0I_B) = (I - \Pi_B{}^t)R^tI_B\,.$$

However,

$$I_B + (I_BR_0I_B) = (I_B - I_B\Pi_B)^{-1}$$

(see proof of Theorem 3.2 (ii) in [2]), and thus, for $y \in B$,

$${}_BR^t(x, y) = [(I - \Pi_B{}^t)R^t(I_B - I_B\Pi_B)](x, y). \tag{2.7}$$

Consequently, for $y \in B$,

$$
\begin{aligned}
{}_BR_n(x, y) &= [R_n I_B(I - \Pi_B)](x, y) \\
&\quad - \sum_{k=1}^{n} \sum_{z \in B} {}_BP_k(x, z)\,[R_{n-k}(I_B - I_B\Pi_B)]\,(z, y).
\end{aligned} \tag{2.8}
$$

Hence for $y \in B$

$$
\begin{aligned}
\frac{\sum_{j=0}^{n} {}_BR_j(x, y)}{\sum_{j=0}^{n} R_j} &= \frac{\sum_{z \in B} \sum_{j=0}^{n} R_j(x, z)\,[(I_B - I_B\Pi_B)]\,(z, y)}{\sum_{j=0}^{n} R_j} \\
&\quad - \sum_{z,u \in B} \left\{ \frac{\sum_{j=1}^{n} \left[\sum_{k=1}^{j} {}_BP_k(x, z) \right] R_{n-j}(z, u)}{\sum_{j=0}^{n} R_j(z, u)} \; \frac{\sum_{j=0}^{n} R_j(z, u)}{\sum_{j=0}^{n} R_j} \right\} \\
&\qquad \times [(I_B - I_B\Pi_B)](u, y).
\end{aligned}
$$

Since

$$
\frac{R_n(z, u)}{\sum_{j=0}^{n} R_j(z, u)} \to 0
$$

and, for $z \in B$,

$$
\sum_{k=1}^{n} {}_BP_k(x, z) \to {}_BR_0(x, z) = \Pi_B(x, z),
$$

we obtain from the above expression, by appeal to a well-known result on Norlünd summation, that (2.2) holds. Now for an arbitrary y, we obtain from (2.6) that

$$
\begin{aligned}
{}_BR_n(x, y) = R_n(x, y) &- \sum_{k=1}^{n} \sum_{z \in B} {}_BP_k(x, z) R_{n-k}(z, y) \\
&- \sum_{z \in B} {}_BR_n(x, z) R_0(z, y).
\end{aligned}
$$

Using this identity and a summation argument very similar to that used to establish (2.2) from (2.8), we obtain (2.3). Finally, if the chain is strongly transient, then

$$
\sum_{n} {}_BR_n(x, y) < \infty,
$$

and, for $y \in B$,

$$
{}_BR_n(x, y) = \sum_{z} {}_BP_{n+1}(x, z) H_B(z, y).
$$

Consequently,

$$\sum_{n=0}^{\infty} {}_BR_n(x, y) = \sum_z {}_BR_0(x, z)H_B(z, y),$$

which establishes (2.4). This completes the proof.

The results of Theorem 2.1 can be used to give interesting results on the time-dependent Dirichlet problem for a finite set B. This problem is the following: Find a sequence $\{\varphi_n(x)\}$ of functions such that

$$\begin{aligned} (P - I)\varphi_n(x) &= (\varphi_{n+1} - \varphi_n)(x), && x \notin B,\ n > 0; \\ \varphi_n(x) &= f(x), && x \in B,\ n \geq 0; \\ \varphi_0(x) &= 0, && x \notin B,\ n = 0. \end{aligned}$$

These equations have a unique solution, which is

$$\begin{aligned} \varphi_n(x) &= \sum_{y \in B} P_x(V_B \leq n;\ X_{V_B} = y)f(y), && x \notin B, \\ &= f(x), && x \in B. \end{aligned}$$

As $n \to \infty$,

$$\varphi_n(x) \to \varphi(x) = \sum_{y \in B} H_B(x, y)f(y),$$

which is a solution (but not the only solution) of the time-independent Dirichlet problem for the same set B. That is, $\varphi(x)$ is bounded and

$$\begin{aligned} (I - P)\varphi(x) &= 0, && x \notin B, \\ \varphi(x) &= f(x), && x \in B. \end{aligned}$$

But then,

$$\begin{aligned} 0 \leq \varphi(x) - \varphi_n(x) &= \sum_{y \in B} P_x(n < V_B < \infty;\ X_{V_B} = y)f(y), && x \notin B, \\ &= 0, && x \in B. \end{aligned}$$

Thus from Theorem 2.1 we obtain the following.

COROLLARY 2.1. *In a regular chain,*

$$\lim_{m \to \infty} \frac{\sum_{n=0}^{m} | \varphi(x) - \varphi_n(x) |}{\sum_{n=0}^{m} R_n(0, 0)} = [(I - \Pi_B)A\Pi_B(I - \Pi_B)f](x), \qquad x \notin B. \tag{2.9}$$

In particular, in every strongly transient chain,

$$\sum_{n=0}^{\infty} | \varphi(x) - \varphi_n(x) | = {}_BR_0 H_B f(x), \qquad x \notin B. \tag{2.10}$$

We will now establish the main result of this section.

THEOREM 2.2. *In a regular transient chain, for $r \geq 1$ and $y \in B$,*

$$\lim_{m\to\infty} \frac{\sum_{n=0}^{m} P_x(n < V_B^r < \infty;\ X_{V_B^r} = y)}{\sum_{n=0}^{m} R_n} = L_B(x, y; r), \tag{2.11}$$

where

$$L_B(x, y; r) = \left[\sum_{j=0}^{r-1} (\Pi_B)^j (M_B I_B)(\Pi_B)^{r-1-j} \right] (x, y). \tag{2.12}$$

In particular, if the chain is strongly transient,

$$\sum_{n=0}^{\infty} P_x(n < V_B^r < \infty;\ X_{V_B^r} = y) = \left[\sum_{j=0}^{r-1} (\Pi_B)^j ({}_BR_0 H_B) \Pi_B^{r-1-j} \right] (x, y). \tag{2.13}$$

PROOF: A simple induction argument shows

$$\sum_{n=1}^{\infty} P_x(V_B^r = n;\ X_n = y) t^n = (\Pi_B^t)^r(x, y).$$

Set

$${}_BR_r^t(x, y) = \sum_{n=0}^{\infty} t^n P_x(n < V_B^r < \infty;\ X_{V_B^r} = y).$$

Then, by an easy computation,

$${}_BR_r^t(x, y) = (\Pi_B^t\, {}_BR_{r-1}^t)(x, y) + ({}_BR_1^t (\Pi_B)^{r-1})(x, y). \tag{2.14}$$

Let ${}_BR_r(x, y, n) = P_x(n < V_B^r < \infty;\ X_{V_B^r} = y)$. Then from the above equation, we obtain

$$\begin{aligned} {}_BR_r(x, y, n) = \sum_{u \in B} \Bigg[& \sum_{j=1}^{n} {}_BP_j(x, u)\, {}_BR_{r-1}(u, y, n - j) \\ & + {}_BR_1(x, u, n)\, (\Pi_B)^{r-1}(u, y) \Bigg]. \end{aligned} \tag{2.15}$$

Hence

$$\frac{\sum_{n=0}^{m} {}_BR_r(x, y, n)}{\sum_{n=0}^{m} R_n} = \sum_{u \in B} \left| \frac{\sum_{n=0}^{m} {}_BR_{r-1}(u, y, m-n) \sum_{j=1}^{n} {}_BP_j(x, u)}{\sum_{n=0}^{m} {}_BR_{r-1}(u, y, n)} \right.$$
$$\times \left. \frac{\sum_{n=0}^{m} {}_BR_{r-1}(u, y, n)}{\sum_{n=0}^{m} R_n} + \frac{\sum_{n=0}^{m} {}_BR_1(x, u, n)\,(\Pi_B)^{r-1}(u, y)}{\sum_{n=0}^{m} R_n} \right| \tag{2.16}$$

We now proceed by induction on r. For $r = 1$, the assertions of the theorem are just those of Theorem 2.1 and thus the theorem holds for $r = 1$. Assume that we have already established that the limit in (2.11) exists for all values up to $r - 1$ and that (2.12) holds for all these values. Then, by (2.16) and a well-known result on Norlünd means, we see that the limit in (2.11) exists for r and satisfies the equation

$$\begin{aligned} L_B(x, y; r) = &\sum_{u \in B} \Pi_B(x, u) L_B(u, y; r-1) \\ &+ \sum_{u \in B} M_B(x, u)\,(\Pi_B)^{r-1}(u, y), \qquad r > 1 \end{aligned} \tag{2.17}$$

with the initial condition,

$$L_B(x, y; 1) = M_B(x, y).$$

The unique solution of this equation is (2.12). Equation (2.13) follows at once from Eqs. (2.11) and (2.12) by (2.4).

Recalling the fact that

$$[V_B^{r+1} > n] = [N_n(B) \leq r],$$

we see that

$$P_x(n < V_B^r < \infty) = P_x(N_n(B) < r) - P_x(N(B) < r).$$

Hence we at once obtain the following result from Theorem 2.2.

COROLLARY 2.2. *In a regular transient chain*

$$\lim_{m \to \infty} \frac{\sum_{n=1}^{m} [P_x(N_n(B) \leq r) - P_x(N(B) \leq r)]}{\sum_{n=0}^{m} R_n} = \tag{2.18}$$

$$\left[\sum_{j=0}^{r} (H_B)^j (M_B I_B)\, (H_B)^{r-j} 1_B \right] (x).$$

In particular, in every strongly transient chain,

$$\sum_{n=1}^{\infty} [P_x(N_n(B) \leq r) - P_x(N(B) \leq r)] = \left[\sum_{j=0}^{r} (H_B)^j {}_B R_0 H_B (H_B)^{r-j} 1_B \right] (x) \tag{2.19}$$

For the special case of a recurrent event these results take an especially simple form. In this case we may establish these results with no assumptions whatsoever.

THEOREM 2.3. *In every transient recurrent event*

$$\lim_{m\to\infty} \frac{\sum_{n=0}^{m} [P(N_n \leq r) - P(N \leq r)]}{\sum_{n=0}^{m} R_n} = (r+1)\varrho^r(1-\varrho)^2, \tag{2.20}$$

where $\varrho = P(V < \infty)$ *(i.e.,* $\varrho =$ *probability that the event takes place),* N_n *is the number of occurrences by time* n, $N = \lim_n N_n$, $R_n = \sum_{j=n+1}^{\infty} u_j$ *and* $u_n = EN_n - EN_{n-1}$, $(u_0 = 1)$. *In particular, if* $\sum_n R_n < \infty$, *then*

$$\sum_{n=0}^{\infty} [P(N_n \leq r) - P(N \leq r)] = (r+1)\varrho^r(1-\varrho)^2 U'(1), \tag{2.21}$$

where $U(t) = \sum_{n=0}^{\infty} u_n t^n$.

PROOF: These results may be established by arguments similar to (but simpler than) those used to establish Theorems 2.1 and 2.2, and so we shall omit the proof.

We conclude this section with the following remark.

REMARK. Observe that, for $y \in B$, $P_x(X_n = y,\ N_n(B) = r) = P_x(X_n = y,\ V_B^r = n)$, and thus, for $y \in B$,

$$\sum_{n=m+1}^{\infty} P_x(X_n = y,\ N_n(B) = r) = P_x(m < V_B^r < \infty\ X_{V_B^r} = y). \tag{2.22}$$

Thus the results of Theorem 2.2 yield information about the tails in

(2.22). By arguments similar to those used to establish Theorem 2.2, we may show that in every regular chain, for $y \notin B$,

$$\lim_{n\to\infty} \frac{\sum_{m=0}^{n} \left[\sum_{j=m+1}^{\infty} P_x(X_j = y, N_j(B) = r) \right]}{\sum_{m=0}^{n} R_m}$$

$$= \left[\sum_{j=0}^{r-1} (\Pi_B)^j (M_B I_B)(\Pi_B)^{r-1-j}{}_B R_0 \right](x, y) + [(\Pi_B)^r M_B](x, y).$$

However, we shall omit these details.

3. Strong Ratio Limits

In this section we will show that under much more restrictive conditions stronger versions of the previous results are possible.

Definition 3. A transient, irreducible, Markov chain is completely regular if for any two states x, y,

$$\lim_{n\to\infty} \frac{R_n(x, y)}{R_n} = D(x, y) \tag{3.1}$$

exists, where, as before, $R_n = R_n(0, 0)$.

Remark. Of course every completely regular chain is regular. In the case of a weakly transient chain, $D(x, y) = A(x, y)$, but for a strongly transient chain, these two quantities are generally different.

Theorem 3.1. *If in a completely regular chain we know that*

$$\lim_{n\to\infty} \frac{R_{n+1}}{R_n} = 1 \tag{3.2}$$

and that for some positive real number $\alpha \leq \frac{1}{2}$,

$$\sup_n \frac{R_{[\alpha n]}}{R_n} < \infty, \tag{3.3}$$

then

$$\lim_{n\to\infty} \frac{P_x(n < V_B{}^r < \infty;\ X_{V_B{}^r} = y)}{R_n} = \left[\sum_{j=0}^{r-1} (\Pi_B)^j (M_B{}^* I_B)(\Pi_B)^{r-1-j}\right](x,y) \tag{3.4}$$

where

$$M_B{}^*(x, y) = [(I - \Pi_B) D I_B (I - \Pi_B)](x, y). \tag{3.5}$$

PROOF: We first establish the theorem for $r = 1$. From Eq. (2.8) we see that, for $y \in B$,

$$\frac{{}_B R_n(x, y)}{R_n} = \frac{\sum_{z\in B} R_n(x, z)(I - \Pi_B)(z, y)}{R_n}$$

$$- \frac{\sum_{k=1}^{n} \sum_{z\in B} {}_B P_k(x, z)\,[R_{n-k} I_B\,(I - \Pi_B)]\,(z, y)}{R_n}.$$

Now the first term on the right converges to $[D I_B(I - \Pi_B)](x, y)$. Write the second term as

$$\sum_{k=1}^{m} + \sum_{k=m+1}^{n}$$

and observe that

$$\lim_{m\to\infty} \lim_{n\to\infty} \sum_{k=1}^{m} = [\Pi_B D I_B (I - \Pi_B)]\,(x, y).$$

Consequently, to complete the proof for $r = 1$ we must show that

$$\lim_{m} \overline{\lim_{n\to\infty}} \left|\sum_{k=m+1}^{n}\right| = 0.$$

Now

$$\left|\sum_{k=m+1}^{n}\right| \le 2 \sum_{y,z\in B} \sum_{k=m+1}^{n} {}_B P_k(x, z) R_{n-k}(z, y) R_n^{-1}$$

$$\le 2 \sum_{y,z\in B} \sum_{k=m+1}^{n} P_k(x, z) R_{n-k}(z, y) R_n^{-1},$$

and thus it suffices to show that for each x, y, z.

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \sum_{k=m+1}^{n} P_k(x, z) R_{n-k}(z, y) R_n^{-1} = 0. \tag{3.6}$$

By irreducibility, there is a constant β and an integer $r > 0$ such that

$$\sum_{k=m+1}^{n} P_k(x, z)R_{n-k}(z, y) \leq \beta \sum_{k=m+1}^{n} P_{k+r}(0, 0)R_{n-k}(0, 0).$$

Since

$$\begin{aligned}\sum_{k=m+1}^{n} P_{k+r}(0, 0)R_{n-k}(0, 0) &= \sum_{j=m+1+r}^{n+r} P_j(0, 0)R_{n+r-j}(0, 0)\\ &\leq \sum_{j=m+1}^{n+r} P_j(0, 0)R_{n+r-j}(0, 0),\end{aligned}$$

we see that (3.6) will follow once we establish

$$\lim_{m} \overline{\lim_{n}} \sum_{j=m+1}^{n+r} P_j(0, 0)R_{n+r-j}(0, 0)R_n^{-1} = 0. \tag{3.7}$$

To show this we may proceed as follows. Write

$$\sum_{j=m+1}^{n+r} = \sum_{j=m+1}^{[an]} + \sum_{[an]+1}^{n+r-m} + \sum_{n+r-m+1}^{n+r}$$

Now

$$R_n^{-1} \sum_{j=m+1}^{[an]} \leq \frac{R_{n+r-[an]}}{R_n} \sum_{j=m+1}^{\infty} P_j(0, 0),$$

and thus

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} R_n^{-1} \sum_{j=m+1}^{[an]} = 0.$$

Next, the second sum on the right is dominated by

$$\frac{R_m}{R_n} \sum_{j=[an]+1}^{\infty} P_j(0, 0) = R_m \frac{R_{[an]}}{R_n},$$

and thus

$$\lim_{m} \overline{\lim_{n}} R_n^{-1} \sum_{[an]+1}^{n+r-m} = 0.$$

Finally, the third sum is dominated by

$$\frac{R_0}{R_n} \sum_{n+r-m+1}^{n+r} P_j(0, 0) = \left[\frac{R_{n+r-m} - R_{n+r}}{R_n}\right] R_0$$

and thus

$$\lim_{n\to\infty} \sum_{n+r-m+1}^{n+r} = 0.$$

Hence (3.7) holds, and this establishes (3.4) for the case $r = 1$. Assume we have already established the theorem for all values $\leq r - 1$. From Eq. (2.15) we obtain (in the notation used in the proof of Theorem 2.2) that

$$\frac{{}_BR_r(x, y, n)}{R_n} = \frac{\sum_{u \in B} {}_BR_1(x, u, n)\,(\Pi_B^{r-1})\,(u, y)}{R_n}$$
$$+ \frac{\sum_{u \in B} \sum_{j=1}^{m} {}_BP_j(x, u)\,{}_BR_{r-1}(u, y, n - j)}{R_n}$$
$$+ \frac{\sum_{u \in B} \sum_{j=m+1}^{n} {}_BP_j(x, u)\,{}_BR_{r-1}(u, y, n - j)}{R_n}$$

The first term on the right converges to the limit $M_B{}^*I_B(\Pi_B)^{r-1}(x, y)$. In the second term (by the induction hypothesis),

$$\lim_{m \to \infty} \lim_{n \to \infty} = \left[\Pi_B \left(\sum_{j=0}^{r-2} (\Pi_B)^j (M_B{}^*I_B)\,(\Pi_B)^{r-2-j}\right)\right] (x, y).$$

Since in the third term, ${}_BR_{r-1}(u, y, n - j) \leq R_{n-j}(u, y)$, we see that by (3.6),

$$\lim_{m \to \infty} \overline{\lim_{n \to \infty}} = 0.$$

Hence (3.4) and (3.5) hold for r, and thus by induction for all $r \geq 1$. This completes the proof.

From this result, stronger versions of Corollaries 2.1 and 2.2 follow.

COROLLARY 3.1. *Under the same hypothesis as Theorem* 3.1, *for the Direchlet problem discussed in Section* 2, *we see that*

$$\lim_{n \to \infty} \frac{|\,\varphi(x) - \varphi_n(x)\,|}{R_n} = \sum_{y \in B} M_B{}^*(x, y)\varphi(y), \qquad x \notin B \quad (3.8)$$

COROLLARY 3.2. *Under the same hypothesis as Theorem* 3.1,

$$\lim_{n \to \infty} \frac{[P_x(N_n(B) \leq r) - P_x(N(B) \leq r)]}{R_n} = \left[\sum_{j=0}^{r} (\Pi_B)^j (M_B{}^*I_B)\,(\Pi_B)^{r-j}\right] (x, y). \quad (3.9)$$

Finally, in the case of a recurrent event no regularity assumptions are necessary. Arguments similar to those used to establish Theorem 3.1 then show the following.

THEOREM 3.2. *If in a transient recurrent event we know that* (3.2) *and* (3.3) *hold, then*

$$\lim_{n\to\infty} \frac{[P(N_n \le r) - (1 - \varrho^{r+1})]}{R_n} = (r + 1)\varrho^r(1 - \varrho)^2. \tag{3.10}$$

4. EXAMPLES

In this section we will apply the preceding results to several specific Markov chains. In all these examples it will turn out that, when the chain is weakly transient $A(x, y) = \mu(y)$ where μ is a specific superregular measure. We will also find that, when the chain is completely regular, then $D(x, y) = \mu(y)$. When $A(x, y)$ and/or $D(x, y)$ have the above form, the expressions for the limits in the preceding theorems assume a particularly nice form because, in these cases,

$$M_B(x, y) = P_x(V_B = \infty)\hat{P}_y(V_B = \infty)\mu(y), \tag{4.1}$$

where ^ denotes quantities computed for the μ-reverse chain. This is the chain with transition matrix

$$\hat{P}(x, y) = \mu(y)P(y, x)\mu(x)^{-1}.$$

Using (4.1) we may reformulate Theorem 2.2 in these cases as follows.

THEOREM 4.1. *Assume that the transient chain is regular. Then on the one hand, if it is weakly transient and* (4.1) *holds,*

$$\lim_{m\to\infty} \frac{\sum_{n=0}^{m} P_x(n < V_B{}^r < \infty; X_{V_B{}^r} = y)}{\sum_{n=0}^{m} R_n} \tag{4.2}$$

$$= \sum_{j=0}^{r-1} P_x(N(B) = j)\hat{P}_y(N(B) = r - 1 - j)\mu(y).$$

On the other hand, if it is strongly transient,

$$\sum_{n=0}^{\infty} P_x(n < V_B{}^r < \infty;\ x_{V_B{}^r} = y) = \left[\sum_{j=0}^{r-1} (\Pi_B)^j ({}_BR_0H_B)\,(\Pi_B)^{r-1-j}\right](x, y). \tag{4.3}$$

EXAMPLE 1: *Transient Random Walk.* Let the state space Ω consist of all integer lattice points in d-dimensional Euclidean space. The transition matrix is $P(x, y) = P(0, y - x)$. We always assume that the random walk is irreducible and transient. This example was discussed at length in [2], to which we refer the reader for details. By Corollary 4.2 of [2], every such random walk is completely regular and, moreover, $D(x, y) = \mathbf{1}$, where $\mathbf{1} = (1, 1, ..., 1)$. Also, condition (3.2) holds whenever $\limsup P_n(0, 0)^{1/n} = 1$, and if the partial sums S_n, which constitute the random walk, are such that S_n/b_n converge in law to a stable distribution of exponent α, having non-zero density $f_\alpha(0)$ at the origin, then condition (3.3) is satisfied as well (Theorem 4.5 of [2]). Hence we see that Theorem 4.1 applies to every irreducible, transient random walk and that the sharper Theorem 3.1 applies, as indicated above, to a very large class of such random walks. Observe finally that, for these walks, for the Dirichlet problem of Section 2, we obtain the interesting fact that

$$\lim_{n\to\infty} R_n^{-1} \mid \varphi_n(x) - \varphi(x) \mid = P_x(V_B = \infty)\hat{C}_B f, \qquad x \notin B,$$

where $\hat{C}_B(y) = \hat{P}_y(V_B = \infty)1_B(y)$ is the "dual capacitary measure" of B.

EXAMPLE 2: *Discrete Time Diffusion on the Non-negative Integers.* This process (also called discrete time birth and death processes or random walks) has the non-negative integers for states and the transition matrix is $P(x, x + 1) = p_x$, $P(x, x - 1) = q_x$, $P(x, x) = r_x$, $P(x, y) = 0$ if $| x - y | > 1$. Here $p_x + q_x + r_x = 1$ and, for $x > 0$, $0 < p_x < 1$ and $0 < q_x < 1$. In addition, $q_0 = 0$ and $0 < p_0 \leq 1$. This example was also discussed in [2], where it was shown that every such transient process is regular and, moreover, that $A(x, y) = \mu(y)$, where $\{\mu(y)\}$ is the unique invariant measure of the process with $\mu(0) = 1$. Hence Theorem 4.1 applies to all these processes.

By an argument very similar to that used to establish the corresponding fact for our next example, we can show that these processes are completely regular whenever $\lim_n R_{n+1}/R_n = 1$. In particular, if for some slowly varying function $L(\cdot)$ and $\alpha > 0$ we know that $R_n \sim n^{-\alpha}L(n)$, then we may conclude that Theorem 3.1 holds.

EXAMPLE 3: *Slowly Spreading Chains of the First Kind.* In [1], a very useful class of chains was introduced by Kemeny. These chains all have the non-negative integers for states, and their transition matrix is of the form

$$\begin{aligned} P(x, y) &= b_y P(x, 0), && y \leq x, \\ &= p_y, && y = x + 1, \\ &= 0, && y > x + 1. \end{aligned}$$

Here $0 < p_x < 1$, $0 \leq b_y < 1$, $b_0 > 0$, and $P(x, 0) > 0$. These chains are always irreducible. According to [1], the only non-negative regular functions for these chains are constants. Set

$$\beta_0 = 1, \ \beta_i = \prod_{k=1}^{i} p_k, \qquad i > 0,$$

and set $\sigma_0 = 0$,

$$\sigma_i = \sum_{k=0}^{i-1} b_k, \qquad i > 0.$$

Then the chains is transient [1] if and only if

$$\sum_{k=1}^{\infty} (1 - p_k)(\beta_k \sigma_k)^{-1} < \infty.$$

Using results in [1], we find that the chain is strongly transient if and only if

$$\sum_{n=0}^{\infty} R_n = \sum_{x} (1 + \alpha_x S_x) S_x < \infty,$$

where $\alpha_x = \beta_x \sigma_{x+1}$, and

$$S_x = \sum_{k=x+1}^{\infty} (1 - p_k)(\beta_k \sigma_k)^{-1}.$$

PROPOSITION 4.1. A transient slowly spreading chain of the first kind is always regular.

PROOF: Since every strongly transient chain is regular, we may assume the chain is weakly transient. From Lemma 3.1 of [2], we know that for each x, y there are constants $\alpha(x, y)$, $\beta(x, y)$ such that

$$0 < \alpha \leq \frac{R_n(x, y)}{R_n} \leq \beta < \infty \tag{4.4}$$

and thus there is a subsequence $\{n_i\}$ such that

$$\lim_{i\to\infty} \frac{\sum_{j=0}^{n_i} R_j(x, y)}{\sum_{j=0}^{n_i} R_j} = A(x, y) \tag{4.5}$$

exists for all x, y. Let

$$S_n(x, y) = \sum_{j=0}^{n} R_j(x, y).$$

Then

$$\sum_{u=0}^{x+1} P(x, u) S_{n_i}(u, y) = S_{n_i}(x, y) + \varepsilon_{n_i}(x, y),$$

where $|\varepsilon_{n_i}| \leq 2\beta R_0$. Hence

$$\sum_{u=0}^{x+1} P(x, u) A(u, y) = A(x, y).$$

Thus for each fixed y, $A(\cdot, y)$ is a non-negative regular function, and consequently for every x, $A(x, y) = A(0, y)$. But we also have for any $y > 0$,

$$\sum_{u=y-1}^{\infty} S_{n_i}(x, u) P(u, y) = S_{n_i}(x, y) + \varepsilon_{n_i}(x, y);$$

or, in other words, for $y > 0$,

$$S_{n_i}(x, y - 1) p_y + b_y \sum_{u=y}^{\infty} S_{n_i}(x, u) P(u, 0) = S_{n_i}(x, y) + \varepsilon_{n_i}. \tag{4.6}$$

For $y = 0$,

$$\sum_{u=0}^{\infty} S_{n_i}(x, u)P(u, 0) = S_{n_i}(x, 0) + \delta_{n_i}, \tag{4.7}$$

where $|\delta_{n_i}| \leq 2\beta R_0$. From (4.6) and (4.7) we obtain

$$S_{n_i}(x, y-1)p_y + b_y \left[S_{n_i}(x, 0) - \delta_{n_i} - \sum_{u=0}^{y-1} S_{n_i}(x, u)P(u, 0)\right] = S_{n_i}(x, y) + \varepsilon_{n_i}.$$

Passing to limits we see that

$$A(x, y-1)p_y + b_y \left[A(x, 0) - \sum_{u=0}^{y-1} A(x, u)P(u, 0)\right] = A(x, y).$$

For each fixed x, this determines $A(x, \cdot)$ recursively; and, by results in [1], we see that $A(x, y) = A(x, 0)a_y$. Consequently, $A(x, y) = A(x, 0)a_y = A(0, 0)a_y = a_y$ (since $A(0, 0) = 1$). Finally, if there were another subsequence $\{n_i\}$ such that the limits in (4.5) held, then the same argument as above would show its value to be a_y. Hence

$$\lim_{n\to\infty} \frac{S_n(x, y)}{S_n(0, 0)} = a_y,$$

and thus Theorem 4.1 applies to every such chain.

In general these chains need have no invariant measure, so $\{a_y\}$ will not be an invariant measure. In fact (see [1]), $\{a_y\}$ will be invariant if and only if $\lim_{n\to\infty} \beta_n = 0$. Moreover, these chains have a unique (modulo a constant) invariant measure if they have one at all, so $\{a_y\}$ will be the only such measure.

PROPOSITION 4.2. In a slowly spreading chain of the first kind,

$$\lim_{n\to\infty} \frac{R_{n+m}(x, y)}{R_n(0, 0)} = a_y \tag{4.8}$$

for arbitrary x, y and all non-negative integers m if and only if

$$\lim_{n\to\infty} (R_{n+1}/R_n) = 1.$$

PROOF. If (4.8) holds, then setting $x = y = 0$, $m = 1$ yields $R_{n+1} \sim R_n$. Now assume that $R_{n+1} \sim R_n$. Since

$$\frac{P_{n+1}(x, y)}{R_n} = O\left(\frac{P_{n+r}(0, 0)}{R_n}\right),$$

we see that

$$\lim_n \frac{P_{n+1}(x, y)}{R_n} = 0.$$

But

$$PR_n = R_n P = R_n - P_{n+1};$$

and by (4.4) there is a subsequence n_i such that

$$\lim_{n\to\infty} \frac{R_{n_i}(x, y)}{R_{n_i}} = D(x, y)$$

for all x, y. Proceeding as in the previous proof we may conclude that $D(x, y) = \alpha_y$. Hence (4.8) holds.

We conclude this section by illustrating the results on recurrent events.

EXAMPLE 4. *Ladder Points of Partial Sums.* Let $\{Y_n, n \geq 1\}$ be a sequence of indepedent, identically distributed, random variables having partial sums $S_n = \sum_{i\leq n} Y_i$. An index n is called a ladder point for the partial sums if $S_n > S_i$ for $i = 0, 1, 2, ..., n$, where $S_0 = 0$. Ladder points are evidently a recurrent event, and

$$u_n = P(S_n > S_i, 0 \leq i < n) = P(S_i > 0, 1 \leq i \leq n).$$

A well-known identity of Sparre-Andersen (see for example, Eq. (1.1) of [3]) asserts that

$$U(t) = \sum_{n=0}^{\infty} u_n t^n = \exp \sum_{k=1}^{\infty} \frac{t_k}{k} P(S_k > 0). \tag{4.9}$$

Consequently, ladder points are transient if and only if

$$\sum_{k=1}^{\infty} P(S_k > 0)k^{-1} < \infty$$

and are strongly transient if and only if

$$\sum_{k=1}^{\infty} P(S_k > 0) < \infty.$$

In particular if $E \mid Y_i \mid < \infty$ and $EY_i < 0$, then ladder points are transient. Moreover, as

$$\sum_{k=1}^{\infty} P(S_k > 0)$$

is dominated by the renewal function at 0 for the negatives of the S_n, we obtain by known results (see, for example, Theorem 1 of [4]) that the event is strongly transient whenever $E \mid Y_i \mid^2 < \infty$, and $EY_i < 0$. But there are also weakly transient ladder points. For example, if $Y_i = Z_i - \mu$, where $\mu > 0$ and Z_i has a symmetric stable distribution of exponent α, $1 < \alpha < 2$, then, for some $A > 0$,

$$\begin{aligned} P(S_n > 0) &= P(Z_1 + \cdots + Z_n) > n\mu) \\ &= P(Z_1 > n^{1-(1/\alpha)}\mu) \sim An^{1-\alpha}. \end{aligned} \tag{4.10}$$

Hence such ladder points are weakly transient.

Using well-known Abelian theorems, we readily obtain from (4.10) that

$$A(t) = \sum_{n=1}^{\infty} t^n P(S_n > 0) \sim A\Gamma(2 - \alpha)\,(1 - t)^{-(2-\alpha)}, \qquad t \to 1^-,$$

and that

$$\int_t^1 A(s)\,ds \sim A\Gamma(2 - \alpha)(1 - t)^{\alpha-1}(\alpha - 1)^{-1}, \qquad t \to 1^- \tag{4.11}$$

Observe that

$$\sum_{k=1}^{\infty} \frac{P(S_k > 0)}{k} t^k = \int_0^t A(s)\,ds;$$

thus from (4.9) we see that as $t \to 1^-$,

$$\begin{aligned} U(1) - U(t) = U(t)\left\{1 - \exp\left[-\int_t^1 A(s)ds\right]\right\} &\sim U(1)\int_t^1 A(s)ds \\ &\sim U(1)A\Gamma(2 - \alpha)\,(1 - t)^{\alpha-1}(\alpha - 1)^{-1}. \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} R_n t^n = \frac{U(1) - U(t)}{1 - t} \sim U(1)A\Gamma(2 - \alpha)(1 - t)^{\alpha-2}(\alpha - 1)^{-1}.$$

Karamata's Tauberian theorem now yields

$$\sum_{k=0}^{n} R_k \sim \frac{U(1)An^{2-\alpha}}{(2 - \alpha)(\alpha - 1)},$$

and, since R_n is monotone, another Tauberian theorem yields the result that

$$R_n \sim \frac{U(1)An^{1-\alpha}}{(\alpha - 1)}$$

Consequently, conditions (3.2) and (3.3) are satisfied in this case.

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